# Unitarity of the tree approximation to the Glauber AA amplitude for large A

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**Abstract**

The nucleus-nucleus Glauber amplitude in the tree approximation is studied for heavy participant nuclei. It is shown that, contrary to previous published results, it is not unitary for realistic values of nucleon-nucleon cross-sections.

## 1 Introduction

Scattering on the nuclei is commonly studied in the Glauber approximation, which can be rigorously derived in Quantum Mechanics provided the transverse momenta transferred to the projectile are much smaller than its longitudinal momentum. With certain reservations it can be generalized to the high energy region where the elementary nucleon-nucleon (NN) amplitudes become predominantly inelastic. For the nucleon-nucleus (NA) scattering the Glauber approximation has a transparent probabilistic interpretation. If the target nucleus is heavy, with atomic number $A >> 1$, the Glauber formula acquires a simple eikonal form, which clearly shows that the resulting amplitude is unitary, that is its modulus is smaller than unity at fixed impact parameter.

With the advent of collider experiments nucleus-nucleus (AB) scattering becomes an important physical object. The Glauber approximation can be easily generalized to the AB case and it was in fact done very long ago. The Glauber formula for AB scattering looks very similar to the hA case. At fixed impact parameter $b$ the scattering matrix $S$ is assumed to be a product of nucleon-nucleon scattering matrices $s$ averaged over the transverse distributions of nucleons in both nuclei:

$$S(b) = \Big\langle \prod_{i=1}^{A}\prod_{k=1}^{B} s(b - x_i + x'_k)\Big\rangle_{A,B}, \tag{1}$$

where $x_i$ and $x'_k$ are the transverse coordinates of the nucleons in the projectile and target nuclei respectively and in absence of correlations in both nuclei averaging $< ... >_{A,B}$ means

$$\Big\langle F(x_i, x'_k)\Big\rangle_{A,B} = \int \prod_{i}^{A}\prod_{k}^{B} d^2x_i d^2x'_k \Big(T_A(x_i)T_B(x'_k - b)F(x_1, ...x_A, x'_1, ...x'_B)\Big). \tag{2}$$

Here $T_A(x)$ and $T_B(x')$ are the standard nuclear profile functions normalized to unity. However in contrast to the NA case the content of the Glauber formula for AB scattering

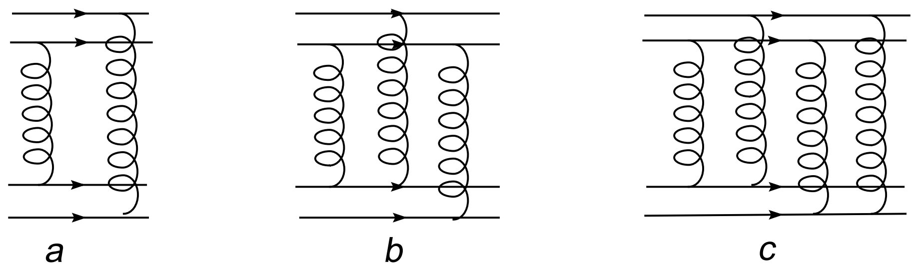


Figure 1: Examples of disconnected ($a$), tree ($b$) and loop ($c$) diagrams for the Glauber AB amplitude with $A = B = 2$

turns out to be much more complicated. Presenting in the standard manner the NN scattering matrix

$$s(b) = 1 + ia(b),$$

where $a$ is the NN scattering amplitude, one obtains from (1) a set of terms corresponding to different ways the nucleons from the projectile and target may interact with each other. Each of these terms may be illustrated by simple diagrams indicating these interaction. Some examples are shown in Fig. 1. for two pairs of interacting nucleons in the projectile and target. One observes that in contrast to hA case the diagrams may contain disconnected parts (Fig.1.$a$) and, most important, loops (Fig.1.$c$), which involve internal integrations over transferred transverse momenta and thus NN amplitudes for non-zero transferred momenta.

Loop contributions depend not only on the total NN cross-sections as the tree diagrams but also on the form of the differential NN cross-section. Their calculation is difficult and unreliable, so that in most applications to heavy nuclei ($A, B >> 1$) loop contributions are simply neglected. The typically used approximation is the so-called optical approximation, which corresponds to taking into account only the simplest contribution (a single NN interaction) for each connected part. In the optical approximation

$$S^{opt}(b) = e^{F^{opt}(b)}, \tag{3}$$

where the optical eikonal $F^{opt}(b)$ is

$$F^{opt}(b) = iaAB \int d^2x T_A(x) T_B(b - x). \tag{4}$$

Its advantage is simplicity and obvious unitarity. The natural question, which has been long discussed in literature, is the quality of the approximation which neglects loops (the tree approximation) for the case $A, B >> 1$ and in particular its relation to the optical approximation. In [1] in the limit $A, B >> 1$ a closed formula was obtained for the tree approximation to the Glauber amplitude, different from the optical approximation but also unitary in the above mentioned sense, that is with a modulus smaller than unity at fixed impact parameter. Their result gives the tree eikonal in the form

$$F^{[1]}(b) = \frac{i}{a} \int d^2x f(\gamma_A, \gamma_B),$$

where

$$\gamma_A(x) = -iaAT_A(x), \quad \gamma_B(x) = -iaBT_B(b-x),$$

$$f(\gamma_A, \gamma_B) = \sum_{l=1}^{k}(-1)^{l+1}(u_l + v_l + u_l v_l) - \gamma_A - \gamma_B$$

and $u_l$ and $v_l$ are the $k$ solutions of the transcendental equations

$$u = \gamma_A e^{-v},$$
$$v = \gamma_B e^{-u}.$$

The number of solutions may be $k = 1$ or $k = 3$. In the latter case it is assumed that $u_1 > u_2 > u_3$ and $v_1 < v_2 < v_3$. One can check by numerical studies that the resulting $F(b)$ is always negative, so that $|S(b)| \leq 1$. However later in [2] it was claimed that in fact at $A, B >> 1$ the sum of all tree diagrams is exactly given by the optical approximation.

The aim of the present study is to resolve this contradiction for the asymptotic of the tree approximation to the Glauber AB amplitude for large $A$ and $B$. Our results are different from both [1] and [2]. Unfortunately they are also much gloomier. Namely we find that at $A, B >> 1$ the sum of all tree diagrams inevitably becomes non-unitary, so that taking loops into account is absolutely necessary for the physically meaningful amplitude. Thus the optimistic hope that the dependence of the AB amplitude on the behaviour of the NN amplitude at non-zero momentum transfers is hardly probable. [1] is unfortunately not justified. The reason why our results are different from the previous ones lies in the details in which simplifications related to the asymptotic at $A, B >> 1$ are made, in particular in the not completely accurate use of the saddle point method in the previous derivations. As we shall see apart from the saddle points which lead to the quasi-optical approximation there exist others which may give the dominant contribution and destroy unitarity.

Our study will be based on the equation which expresses in a compact manner the sum of all tree diagrams, obtained by one of the authors some 30 years ago [3]. The good quality of this equation is that it is valid for arbitrary finite (and even small) values of $A$ and $B$ and therefore presents an adequate starting point to investigate the asymptotic at large $A$ and $B$. To simplify we shall limit ourselves with the case of collision of two identical nuclei $A = B$. Since the problem we address does not depend on the form of the transverse distribution $T_A(x)$ we further simplify our study by assuming that $T(x)$ does not depend on $x$ inside the nucleus:

$$T_A(x) = \theta(R_A - |x|)\frac{1}{\pi R_A^2} \tag{5}$$

where $R_A = A^{1/3}R_0$ is the radius of the nucleus.

# 2 The tree amplitude for AB scattering in the Glauber approximation

## 2.1 General $A$, $B$ and $b$

In [3] an expression was derived for the sum of all tree diagrams for the $S$ matrix in the Glauber approximation to nucleus-nucleus scattering valid at arbitrary finite atomic

numbers $A$ and $B$ of colliding nuclei. At a given impact parameter $b$

$$S(b)=\frac{A!B!}{4\pi^2 i^{A+B+2}}\oint\frac{d\tau d\tau'}{\tau^{A+1}\tau'^{B+1}}e^{i(\tau+\tau')-Z(b,\tau,\tau')}. \tag{6}$$

Here

$$Z=-i\int(d^2x)W\Big(i\tau T_A(x),i\tau' T_B(x')\Big) \tag{7}$$

and

$$(d^2x)=d^2xd^2x'\delta^2(b-x+x').$$

As indicated, $a$ is the nucleon-nucleon forward scattering amplitude and $T_A(x)$ and $T_B(x')$ are the nuclear profile functions at transverse coordinates $x$ and $x'$ respectively. $W$ is the effective classical action for the effective quantum theory of two fields $u$ and $v$

$$W(\rho,\rho')=Y\Big(u(\rho,\rho'),v(\rho,\rho')\rho,\rho',\Big),$$

where

$$Y=\frac{1}{a}vu-i\rho(e^{-u}-1)-i\rho'(e^{-v}-1)$$

and $u$ and $v$ satisfy a pair of transcendental equations

$$u=-ia\rho' e^{-v},\quad v=-ia\rho e^{-v}. \tag{8}$$

Here $\rho=i\tau T_A(x)$ and $\rho'=i\tau' T_B(x')$.

A detailed derivation of this formula can be found in [3]. For convenience it is briefly reproduced in Appendix. In this section for the simplified case of constant $T_A$ and $T_B$ inside the nucleus we transform this formula to the form suitable for our analysis at large $A$ and $B$.

It is trivial to see that $W$ is different from zero only in the overlap region. Indeed for $T_A=0$ and so $\rho=0$ we have $v=0$ and for $T_B=0$ and so $\rho'=0$ we have $u=0$. In both cases $Y=0$. Therefore integration over $x$ and $x'$ is extended over the overlap region only. For constant $T_{A,B}$ it gives precisely the area of the overlap region $G(b)$. Taking this into account and expressing $W$ via $Y$ we find

$$S(b)=\frac{A!B!}{4\pi^2 i^{A+B+2}}\oint\frac{d\tau d\tau'}{\tau^{A+1}\tau'^{B+1}}e^{i(\tau+\tau')}e^{i\frac{G(b)}{a}(vu+\kappa\tau(e^{-u}-1)+\kappa'\tau'(e^{-v}-1)}, \tag{9}$$

where

$$\kappa=\frac{a}{\pi R_A^2},\quad \kappa'=\frac{a}{\pi R_B^2} \tag{10}$$

and equations (8) become

$$u=\kappa'\tau' e^{-v},\quad v=\kappa\tau e^{-u}. \tag{11}$$

Separating in the exponent in (9) the terms proportional to $\tau$ or $\tau'$ we rewrite Eq. (9) as

$$S(b)=\frac{A!B!}{4\pi^2 i^{A+B+2}}\oint\frac{d\tau d\tau'}{\tau^{A+1}\tau'^{B+1}}e^{i\tau\left(1-\frac{G(b)}{\pi R_A^2}\right)}e^{i\tau'\left(1-\frac{G(b)}{\pi R_B^2}\right)}e^{i\frac{1}{a}G(b)(uv+u+v)}. \tag{12}$$

To avoid solving transcendenal equations (11) we pass in (12) to the integration over $u$ and $v$, since it is trivial to express $\tau$ and $\tau'$ via $u$ and $v$ from (11) but not *vice versa*. To do

this we have to find the Jacobian. We denote $\xi = \kappa\tau$ and $\eta = \kappa'\tau'$ Direct differentiation of (11) gives

$$\frac{\partial u}{\partial \xi} = -u\frac{\partial v}{\partial \xi}, \quad \frac{\partial u}{\partial \eta} = \frac{u}{\eta} - u\frac{\partial v}{\partial \eta}, \quad \frac{\partial v}{\partial \xi} = \frac{v}{\xi} - v\frac{\partial u}{\partial \xi}, \quad \frac{\partial v}{\partial \eta} = -v\frac{\partial u}{\partial \eta}.$$

From these equation we immediately obtain

$$\frac{\partial u}{\partial \eta} = \frac{u}{\eta(1-uv)}, \quad \frac{\partial v}{\partial \xi} = \frac{v}{\xi(1-uv)}, \quad \frac{\partial u}{\partial \xi} = -\frac{uv}{\xi(1-uv)}, \quad \frac{\partial v}{\partial \eta} = -\frac{uv}{\eta(1-uv)}.$$

As a result we find the Jacobian

$$J = \frac{\partial(u,v)}{\partial(\xi,\eta)} = -\frac{uv}{\xi\eta(1-uv)}.$$

At small $x$ and $y$ we obviously have $u \sim \eta$ and $v \sim \xi$. So choosing the initial contours in $x$ and $y$ around the origin small enough we find that integrations over $u$ and $v$ will go also around small contours around the origin, which can then be transformed unless we come across some singularities in $u$ and $v$. Expressing $\tau$ and $\tau'$ in terms of $u$ and $v$ as

$$\tau = \frac{1}{\kappa}ve^{u}, \quad \tau' = \frac{1}{\kappa'}ue^{v}$$

we transform Eq. (12) into

$$S(b) = \frac{A!B!}{4\pi^2 i^{A+B+2}}\kappa^A\kappa'^B \oint \frac{dudv}{u^{A+1}v^{B+1}}(1-uv)e^{-Au-Bv}$$
$$e^{i\frac{1}{\kappa}ve^u\left(1-\frac{G(b)}{\pi R_A^2}\right)} e^{i\frac{1}{\kappa'}ue^v\left(1-\frac{G(b)}{\pi R_B^2}\right)} e^{i\frac{1}{a}G(b)(uv+u+v)}. \tag{13}$$

This formula is the starting point for our investigation.

## 2.2 Case $A = B$ and $b = 0$

Our formula for $S(b)$ greatly simplifies in the case of central collisions of identical nuclei, when $A = B$ and $b = 0$. In this case the complicated exponents in the first two exponentials in (13) are absent and we find a simple expression

$$S(0) = \frac{(A!)^2}{4\pi^2 i^{2A+2}}\kappa^{2A} \oint \frac{dudv}{(uv)^{A+1}}(1-uv)e^{-A(u+v)}e^{-\frac{1}{i\kappa}(uv+u+v)}. \tag{14}$$

It is straightforward to find this $S$ matrix in the form of a finite sum of terms. Integrations over $u$ and $v$ obviously give the coefficient before term $(uv)^A$ in the expansion of the rest part of the integrand in powers of $u$ and $v$ (with factor $(2\pi i)^2$ which cancels with the analogous factor in front of the whole expression in (14)). So our problem reduces to the expansion in powers of $u$ and $v$ of the three exponentials in (14). We find

$$e^{\left(-\frac{1}{i\kappa}-A\right)u}e^{\left(-\frac{1}{i\kappa}-A\right)v}e^{-\frac{1}{i\kappa}uv}$$

$$= \sum_{n_1,n_2,n_3} \frac{1}{n_1!n_2!n_3!}\left(-\frac{1}{i\kappa}\right)^{n_3}\left(-\frac{1}{i\kappa}-A\right)^{n_1+n_2} u^{n_1+n_3}v^{n_2+n_3} =$$

$$= \sum_{n=0} \frac{1}{n!}\left(-\frac{1}{i\kappa}\right)^n \sum_{n_1,n_2\geq n} \frac{u^{n_1}v^{n_2}}{(n_1-n)!(n_2-n)!}\left(-\frac{1}{i\kappa}-A\right)^{n_1+n_2-2n}. \tag{15}$$

Integration over $u$ and $v$ gives the term with $n_1 = n_2 = A$. Without the factor $(2\pi i)^2$ and the one in front of the whole expression (14) it is

$$\sum_{n=0}^{A} \frac{1}{n![(A-n)!]^2}\left(-\frac{1}{i\kappa}\right)^n \left(-\frac{1}{i\kappa}-A\right)^{2(A-n)}$$

$$= \sum_{n=0}^{A} \frac{1}{n![(A-n)!]^2}\left(-\frac{1}{i\kappa}\right)^{2A-n} (1-\gamma)^{2(A-n)}. \tag{16}$$

where we defined

$$\gamma = -iA\kappa. \tag{17}$$

From this expression one has to subtract the second one which comes from the term $-uv$ in the Jacobian. Obviously it gives the $(A-1)$th term in the expansion of the integrand in powers of $u$ and $v$ and is obtained from (16) by the substitution $A \to A-1$.

Collecting all the factors we get

$$S(0) = (A!)^2(1-\gamma)^{2A}\Big\{\sum_{n=0}^{A} \frac{1}{n![(A-n)!]^2}\left(\frac{\gamma}{A(1-\gamma)^2}\right)^n -$$

$$\frac{\gamma^2}{A^2(1-\gamma)^2}\sum_{n=0}^{A-1} \frac{1}{n![(A-1-n)!]^2}\left(\frac{\gamma}{A(1-\gamma)^2}\right)^n\Big\}, \tag{18}$$

Note that in fact the scattering amplitude $a$ is pure imaginary at high energies:

$$a = \frac{i}{2}\sigma,$$

where $\sigma$ is the total cross-section for pp collisions. Threrefore $\gamma$ is positive and so each term in the two sums in (18) is positive. This expression gives a simple closed form for the $S$ matrix for collisions of identical nuclei at $b = 0$. The term with $n = 0$ in the first sum is independent of the scattering amplitude $a$ and equal to unity. The rest terms give the amplitude with factor $i$.

Note that (18) contains powers $(1-\gamma)^n = (1 + Ai\kappa)^n$. This is the origin of difficulties related to AB scattering. A similar formula for NA scattering contains powers $(1 + i\kappa)^n$. Since $\kappa$ is small, of the order $A^{-2/3}$, summation over $n$ does not violate unitarity. In contrast, for the nucleus case $\kappa$ is substituted by $A\kappa$, which grows with $A$ as $A^{1/3}$. As a result factors $(1-\gamma)^n$ grow like $A^{n/3}$ at high $A$ and $n$ and as we shall discover make $|S|$ also grow. Unitarity remains valid only for values of $\gamma$ just slightly above unity, which is certainly not satisfied at high enough (and physically interesting) $A$.

Eq. (18) makes it feasible to perform numerical calculations of the tree Glauber amplitude, since each of the two sums contains only positive terms. Numerical results demonstrate that $S(0)$ is unitary, that is $|S(0)| < 1$, at any value of $A$ only provided $0 < \gamma < 1.42$. The last condition means that

$$\frac{A\sigma}{2\pi R_A^2} < 1.42. \tag{19}$$

To compare, a similar condition for $hA$ scattering

$$\frac{\sigma}{2\pi R_A^2}<1$$

is always satisfied for large $A$. However in our case, with $R_A=R_0A^{1/3}$ (19) reduces to

$$\frac{A^{1/3}\sigma}{2\pi R_0^2}<1.42. \tag{20}$$

With $\sigma\sim 2\pi R_0^2$ it is always violated at physically relevant large $A$.

# 3 The asymptotic of the amplitude at $b=0$ and $A=B\to\infty$

In this section we shall derive analytic asymptotic formulas which agree with our numerical results.

We rewrite (14) as

$$S(0)=\frac{(A!)^2}{4\pi^2 i^2}\frac{\gamma^{2A}}{A^{2A}}\oint dudv\Big(\frac{1}{uv}-1\Big)e^{AP(u,v)}, \tag{21}$$

where

$$P(u,v)=-\ln(uv)-u-v+\frac{1}{\gamma}(uv+u+v).$$

We estimate the asymptotic by the saddle point method. The saddle point is determined by the equations

$$\begin{aligned}P_u&=-\frac{1}{u}-1+\frac{1}{\gamma}(v+1)=0,\\ P_v&=-\frac{1}{v}-1+\frac{1}{\gamma}(u+1)=0.\end{aligned} \tag{22}$$

The second derivatives are

$$P_{uu}=\frac{1}{u^2},\quad P_{vv}=\frac{1}{v^2},\quad P_{uv}=\frac{1}{\gamma}.$$

Eqs. (22) have two symmetric solutions

$$u_1=v_1=\gamma,\quad u_2=v_2=-1.$$

**1. Saddle points $u_s=v_s=\gamma$**

Passing to variables $u-\gamma=i\xi$ and $v-\gamma=i\eta$ we rewrite the integral in (21) in the vicinity of the saddle point as

$$S(0)=\frac{(A!)^2}{4\pi^2}\frac{\gamma^{2A}}{A^{2A}}e^{AP(\gamma,\gamma)}\Big(\frac{1}{\gamma^2}-1\Big)\int d\xi d\eta e^{-\frac{A}{2\gamma^2}(\xi^2+2\gamma\xi\eta+\eta^2)}.$$

Subsequent actions depend on the sign of eigenvalues of the matrix

$$M=\frac{A}{2\gamma^2}\left(\begin{array}{cc}1&\gamma\\ \gamma&1\end{array}\right).$$

The eigenvalues are

$$\frac{A}{2\gamma^2}(1+\gamma) \quad \text{and} \quad \frac{A}{2\gamma^2}(1-\gamma).$$

We have to consider two cases $\gamma < 1$ and $\gamma > 1$.

**1.1** Case $\gamma < 1$

In this case we can safely extend the integration regions in both $\xi$ and $\eta$ to the whole real axis to find

$$I = \int d\xi d\eta e^{-\frac{A}{2\gamma^2}(\xi^2+2\gamma\xi\eta+\eta^2)} = \frac{\pi}{\sqrt{\det M}} = \frac{2\pi\gamma^2}{A\sqrt{1-\gamma^2}}. \tag{23}$$

We further use

$$A! = A^A e^{-A}\sqrt{2\pi A} \tag{24}$$

and

$$AP(\gamma,\gamma) = -A\ln\gamma^2 - A\gamma + 2A$$

to finally find

$$S(0) = \sqrt{1-\gamma^2}e^{-A\gamma}. \tag{25}$$

**1.2** Case $\gamma > 1$

In this case we have to rotate one of the variables $\tilde{\xi}$ or $\tilde{\eta}$ which diagonalize matrix $M$ which corresponds to eigenvalue $1-\gamma$ by angle $\pm\pi/2$. Then we get instead of (23)

$$I = \pm i\frac{2\pi\gamma^2}{A\sqrt{\gamma^2-1}}$$

and for $S(0)$

$$S(0) = \mp i\sqrt{\gamma^2-1}e^{-A\gamma} \tag{26}$$

**2. Saddle points** $u_s = v_s = -1$

In this case the prefactor $(uv)^{-1}-1$ vanishes at the saddle point and we have to study it in the vicinity of the saddle point. We put $u = -1-i\xi$ and $v = -1-i\eta$ to find

$$\frac{1}{uv} - 1 = (1-i\xi-\xi^2)(1-i\eta-\eta^2) - 1 = -i\xi - i\eta - \xi^2 - \eta^2 - \xi\eta.$$

Since the leading term vanishes we have to expand the exponent up to terms of the third order in $\xi$ and $\eta$

$$P(u,v) = P(-1,-1) - \frac{1}{2}(\xi^2+\eta^2) - \frac{1}{\gamma}\xi\eta + \frac{1}{3}i(\xi^3+\eta^3),$$

so that in the vicinity of the saddle points the integrand in (21) can be presented as

$$\Big(\frac{1}{uv} - 1\Big)e^{AP(u,v)} = -e^{A(P(-1,-1)-\frac{A}{2}(\xi^2+2\xi\eta/\gamma+\eta^2)}Q(\xi,\eta),$$

where the polynomial $Q(\xi,\eta)$ is

$$Q(\xi,\eta) = \xi^2 + \eta^2 + \xi\eta - \frac{A}{3}(\xi^4+\eta^4+\xi^3\eta+\xi\eta^3).$$

So we find

$$S(0) = -\frac{(A!)^2}{4\pi^2}\frac{\gamma^{2A}}{A^{2A}}e^{AP(-1,-1)}\int d\xi d\eta Q(\xi,\eta)e^{-\frac{A}{2}(\xi^2+2\xi\eta/\gamma+\eta^2)} = -\frac{(A!)^2}{4\pi^2}\frac{\gamma^{2A}}{A^{2A}}e^{AP(-1,-1)}J,$$

where

$$J = j_0 - \frac{A}{3}(j_1 + j_2)$$

with

$$j_0 = -\Big(\frac{\partial}{\partial\alpha} + \frac{1}{2}\frac{\partial}{\partial\beta}\Big)J_0;\quad j_1 = \Big(\frac{\partial^2}{\partial\alpha^2} - \frac{1}{2}\frac{\partial^2}{\partial\beta^2}\Big)J_0;\quad j_2 = \Big(\frac{1}{2}\frac{\partial^2}{\partial\alpha\partial\beta}\Big)J_0,$$

$$J_0 = \int d\xi d\eta e^{-\alpha(\xi^2+\eta^2)-2\beta\xi\eta}$$

and

$$\alpha = \frac{A}{2},\quad \beta = \frac{A}{2\gamma}.$$

In the exponent the matrix in variables $\xi,\eta$ is now

$$M = \begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix}$$

Consider the case $\gamma > 1$ Then $\alpha > \beta$ and both eigenvalues of matrix $M$ are positive. Then we immediately get

$$J_0 = \frac{\pi}{\sqrt{\alpha^2-\beta^2}},\quad j_0 = \frac{2\pi\gamma^2(2\gamma-1)}{A^2(\gamma^2-1)^{3/2}},\quad j_1 = \frac{12\pi\gamma^5}{A^3(\gamma^2-1)^{5/2}},\quad j_2 = -\frac{12\pi\gamma^4}{A^3(\gamma^2-1)^{5/2}}$$

Using also

$$AP(-1,-1) = 2A - \frac{A}{\gamma}$$

and the asymptotic (24) we finally find for $\gamma > 1$

$$S(0) = -\frac{\gamma^2}{A(\gamma+1)^{5/2}(\gamma-1)^{1/2}}e^{A(2\ln\gamma-\frac{1}{\gamma})}. \tag{27}$$

So for $\gamma > 1$ $S(0)$ is always negative and its modulus exponentially grows with $A$ unless $\gamma^2 < \exp(1/\gamma)$, that is $\gamma < \gamma_0 = 1.4215$ when it exponentially falls.

The asymptotic for $\gamma < 1$ requires additional rotation in variables which diagonalize matrix $M$. Up to its sign it can be found just by the analytic continuation from the case $\gamma > 1$. So for $\gamma < 1$

$$S(0) = \pm i\frac{\gamma^2}{A(\gamma+1)^{5/2}(1-\gamma)^{1/2}}e^{A(2\ln\gamma-\frac{1}{\gamma})}. \tag{28}$$

One observes that for $\gamma < 1$ the leading contribution comes from the saddle point $u_s = v_s = \gamma$ and for $\gamma > 1$ from the saddle point $u_s = v_s = -1$, the latter contribution restricting the region of $\gamma > 1$ where unitarity is fulfilled

# 4 Non-central collisions of identical nuclei

We introduce $\gamma$ according to (17) and put

$$G(b) = \lambda(b)\pi R_A^2, \quad 0 \le \lambda \le 1 \tag{29}$$

to rewrite Eq. (13) as

$$S(b) = \frac{(A!)^2\gamma^{2A}}{4\pi^2 i^2 A^{2A}} \oint dudv\Big(\frac{1}{uv} - 1\Big)e^{AP(u,v)}, \tag{30}$$

where now

$$P(u,v) = -\ln(uv) - u - v + \frac{\lambda}{\gamma}(uv + u + v) + \frac{1-\lambda}{\gamma}\Big(ve^u + ue^v\Big).$$

The derivatives are

$$P_u = -\frac{1}{u} - 1 + \frac{\lambda}{\gamma}(v+1) + \frac{1-\lambda}{\gamma}\Big(ve^u + e^v\Big),$$

$$P_v = -\frac{1}{v} - 1 + \frac{\lambda}{\gamma}(u+1) + \frac{1-\lambda}{\gamma}\Big(ue^v + e^u\Big),$$

$$P_{uu} = \frac{1}{u^2} + \frac{1-\lambda}{\gamma}ve^u, \quad P_{uv} = \frac{\lambda}{\gamma} + \frac{1-\lambda}{\gamma}\Big(e^u + e^v\Big).P_{vv} = \frac{1}{v^2} + \frac{1-\lambda}{\gamma}ue^v,$$

As before we seek for symmetric stationary points $u_s = v_s$. Then we obtain an equation

$$(u_s + 1)\Big(-\frac{1}{u_s} + \frac{\lambda}{\gamma} + \frac{1-\lambda}{\gamma}e^{u_s}\Big) = 0.$$

We have the same solution $u_2 = v_2 = -1$ and a new solution $u_1 = v_1$ which satisfies

$$-\frac{1}{u_1} + \frac{\lambda}{\gamma} + \frac{1-\lambda}{\gamma}e^{u_1} = 0$$

or

$$u_1 = \frac{\gamma}{\lambda + (1-\lambda)e^{u_1}}. \tag{31}$$

The actual value of $u_1$ for given $\lambda$ and $\gamma$ can only be found numerically.

**1. Saddle points $u_s = v_s = u_1$.**

We find at the stationary point

$$P = -2\ln u_1 + 2 - 2u_1 + \frac{\lambda}{\gamma}u_1^2,$$

$$P_{uu} = P_{vv} = \frac{1}{u_1^2} + 1 - \frac{\lambda}{\gamma}u_1, \quad P_{uv} = \frac{2}{u_1} - \frac{\lambda}{\gamma}.$$

The determinant of the quadratic form in $u, v$ in the exponent turns out to be

$$\det M = \frac{A^2}{4}(1 - u_1^2)\Big[\frac{1}{u_1^4} - \Big(\frac{1}{u_1} - \frac{\lambda}{\gamma}\Big)^2\Big].$$

It is positive for $u_1 \leq 1$ and arbitrary $\lambda \leq 1$. So for $u_1 \leq 1$ we find the asymptotic

$$S(b) = \sqrt{\frac{1-u_1^2}{1-u_1^2(1-u_1\lambda/\gamma)^2}}\, e^{-A\left(2u_1 - \frac{\lambda}{\gamma}u_1^2 + 2\ln\frac{u_1}{\gamma}\right)}. \tag{32}$$

If $b = 0$ and $\lambda = 1$ then $u_1 = \gamma$ and this asymptotic passes into (25). The bracket in the exponent in (32) is always positive and diminishes with $\lambda$, which implies that the asymptotic gets less falling with the growth of $b$. At $b = 2R_A$ and $\lambda = 0$, when the nuclei only touch each other, the exponent vanishes, which corresponds to $S(b = 2R_A) = 1$ as it should be.

If $u_1 > 1$ then the asymptotic can be obtained by analytic continuation of (32). It remain to be falling with $A$. However in some regions of $\lambda$ and $u_1$ it becomes pure imaginary.

**2. Saddle points $u_s = v_s = -1$.**

At this saddle point we find

$$P = 2 - \frac{\lambda}{\gamma} - \frac{2}{e}\frac{1-\lambda}{\gamma}.$$

Together with the prefactor in (30) it gives an exponential factor in the asymptotic

$$e^{A\left(2\ln\gamma - \frac{\lambda}{\gamma} - \frac{2}{e}\frac{1-\lambda}{\gamma}\right)}. \tag{33}$$

It infinitely grows at $\gamma > \gamma_0(b)$ where

$$2\gamma_0 \ln\gamma_0 = \lambda + \frac{2}{e}(1-\lambda).$$

The value of $\gamma_0(b)$ steadily but slowly grows with $\lambda$ (see Table 1 ). These values restrict the region in which the amplitude remains unitary at $b > 0$.

Table 1. $\gamma_0$ as a function of overlap $\lambda(b)$

| $\lambda$: | 0 | 0.2 | 0.4 | 0.6 | 0.8 | 1.0 |
|---|---|---|---|---|---|---|
| $\gamma_0$: | 1.3211 | 1.3416 | 1.3620 | 1.3820 | 1.4019 | 1.4215 |

Note that at $b = 2R_A$ and thus $\lambda = 0$ the exponential factor (33) grows with $A$ unless $\gamma < \gamma_0(0) = 1.3211$, in spite of the fact that $S(b = 2R_A) = 1$. This seeming contradiction is resolved due to vanishing of the prefactor at exactly $\lambda = 0$. As soon as $\lambda$ gets slightly greater than zero, the $S$-matrix becomes very large and negative for $\gamma > \gamma_0(b)$. For example for $\lambda = 0.001$ and $\gamma = 2.0$ one finds $S(b) = -0.174 \cdot 10^5$, which illustrates that the asymptotic becomes discontinuous at $b = 2R_A$ in the limit of large $A$.

# 5 Wrong ways to study the asymptotic

In this section we illustrate how unwarranted applications of the saddle point method can easily lead to incorrect results for the asymptotic of the tree approxumation to the Glauber AB amplitude, in particular to the results found in refs. [1] and [2].

One may try to study the asymptotic directly from the representation (12) without passing to variables $u$ and $v$. For $A = B$ it can be rewritten as

$$S(b) = \frac{A!^2}{4\pi^2 i^{2A+2}} \oint \frac{d\tau d\tau'}{\tau^{A+1}\tau'^{A+1}} e^{i(1-\lambda)(\tau+\tau')} e^{i\frac{\lambda}{\kappa}(uv+u+v)}, \tag{34}$$

where $u$ and $v$ are determined via $\tau$ and $\tau'$ by Eqs. (11) and $\kappa$ and $\lambda$ are defined by (10) and (29) respectively. The results of [1] are obtained if we separate a factor in the integrand

$$e^{i(\tau+\tau')-A\ln(\tau\tau')}$$

and consider it as the only rapidly changing one at $A \to \infty$. Then the saddle points are

$$i\tau = i\tau' = A.$$

Putting these values in the integrand one obtains the asymptotic of the $S$-matrix in the form

$$S(b) = e^{F(b)},$$

where

$$F(b) = \frac{i\lambda}{\kappa}(uv + u + v) - 2A\lambda$$

and $u$ and $v$ are determined by

$$u = -iA\kappa e^{-v}, \quad v = -iA\kappa e^{-u}.$$

This is the result of [1] for the case of $A = B$ and constant profile functions.

However this derivation is too crude, since it neglects the $A$ dependence of the action. A more elaborate derivation based on variables $u$ and $v$ leads to a pure optical approximation. Introducing for $A = B$ integration variables $t$ and $t'$ defined as

$$i\tau = At, \quad i\tau' = Bt'$$

we find

$$S(b) = \left(\frac{A!}{A^A}\right)^2 \frac{1}{4\pi^2} \int \frac{dt dt'}{tt'} e^{AP(t,t')}, \tag{35}$$

where

$$P_1 = -\ln(tt') + (t + t') + \frac{\lambda}{\gamma}(vu + u + v - r - r')$$

with $r = \gamma t$ and $r' = \gamma t'$, $u$ and $v$ determined via $t$ and $t'$ by equations

$$u = r'e^{-v}, \quad v = re^{-u} \tag{36}$$

and $\gamma$ defined by Eq. (17). Applying the saddle point method to the integral (35) one searches for saddle points from equations

$$\frac{\partial P_1}{\partial t} = \frac{\partial P_1}{\partial t'} = 0.$$

Elementary calculations using Eqs. (36) give

$$\frac{\partial P_1}{\partial t} = 1 - \frac{1}{t} - \lambda + \frac{\lambda}{\gamma}\frac{v}{t}, \quad \frac{\partial P_1}{\partial t'} = 1 - \frac{1}{t'} - \lambda + \frac{\lambda}{\gamma}\frac{u}{t'}$$

If additionally $\lambda = 1$ (central collisions) then the first and third terms cancel and the saddle points are found to be

$$u = v = \gamma.$$

They are the same as we had earlier, after passing to variables $u$ and $v$. They lead to the optical approximation (see (25)). However we do not find the other pair of saddle points $u = v = -1$. The reason is that in variables $t$ and $t'$ this saddle point is transformed into a singularity in the $t, t'$-plane present in the solutions of Eqs. (36). This singularity takes the leading role in the asymptotics at $\gamma > 1$ and leads to the growth of the $S$-matrix and violation of unitarity.

# 6 Conclusions

Using the simplified form of the profile functions, constant inside the colliding nuclei, we have found that the set of tree diagrams for the AB amplitude in the Glauber approximation is not unitary for heavy particpants and realistic values of the NN cross-section. This fact has been found analytically, using the saddle point method in the adequate manner, and fully confirmed by the straightforward numerical calculations. Unitarity is found to be fulfilled only if the NN cross-section $\sigma$ is small and diminishes with $A$ (see Eq. (20)). Previous optimistic results [1, 2] are found to be incorrect due to inadequate application of the saddle point method.

From the practical point of view our results mean that the treatment of the AB scattering in the standard Glauber approximation must inevitably include loop diagrams and so depend on the form of the differential NN cross-section at non-zero angles. It remains to be studied how the situation changes if AB scattering is not described by the Glauber formula but is rather formed by the exchange of self-interacting pomerons (like in the Regge-Gribov model). In this case loops can also be formed, which are usually neglected because formally they are subdominant in the parameter $A^{-1/3}$. However it is not clear if the resulting tree amplitude is unitary when the limitation on the number of interacting nucleons at finite $A$ is correctly imposed. We leave this problem for future studies.

# 7 Acknowledgments

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# 8 Appendix. Derivation of the integral representation for S-matrix.

Consider the contribution to the AB scattering amplitude with a given number $c$ of connected parts and a given number of participant nucleons $n_i$ ($n'_i$) from the projectile (target) in the $i$-th connected part ($i = 1, ..., c$). The standard derivation leads to the expression

$$i\mathcal{A}_{c,n_i,n'_i}(b) = N\frac{A!B!}{c!(A-n)!(B-n')!}(ia)^l\prod_{i=1}^{c}\int d^2x_i T_A^{n_i}(x_i)T_B^{n'_i}(x_i-b). \qquad (37)$$

Here $l$ is the total number of interactions: $l = \sum_{i=1}^{c} l_i$ where $l_i$ is the number of interactions in the $i$-th connected part. Likewise $n = \sum_{i=1}^{c} n_i$ and $n' = \sum_{i=1}^{c} n'_i$ are the total numbers of participants in the projectile and target. In the tree approximation we study $l_i = n_i + n'_i - 1$, so that the number of interactions is uniquely determined by the number of participants. The symmetry factor $N$ arises because with a given number of participant nucleons in each connected part there may be several terms which give identical contribution. To find this factor one may consider an auxiliary zero-dimensional quantum field theory with a generating functional

$$Z = \int D\phi D\phi^\dagger e^{iY(\phi,\phi^\dagger,\rho,\rho')},$$

where

$$Y = a^{-1}\phi^\dagger\phi - i\rho\Big(e^{\phi} - 1\Big) - i\rho'\Big(e^{\phi^\dagger} - 1\Big).$$

This theory will generate the same diagrams as the Glauber expression (37) except that coordinate dependent densities $T_A(x)$ and $T_B(x' - b)$ will be substituted by constants $\rho$ and $\rho'$. The sum of all connected diagrams will be given by the effective action $W$ obtained after integrating out the fields $\phi$ and $\phi^\dagger$. It will depend on powers of $\rho$ and $\rho'$ corresponding to numbers of fields entering in different connected parts

$$iW(\rho,\rho') = \sum_i C_{n_i,n'_i}\rho^{n_i}\rho'^{n'_i}.$$

The coefficients $C_{n_i,n'_i}$ are just the symmetry factors which should be taken into account in Eq. (37) except that they also include the corresponding number of amplitudes $ia$. So we obtain

$$i\mathcal{A}_{c,n_i,n'_i}(b) = \frac{A!B!}{c!(A-n)!(B-n')!}\prod_{i=1}^{c} C_{n_i,n'_i}\int d^2x_i T_A^{n_i}(x_i)T_B^{n'_i}(x'\_b).$$

Since we are interested only in tree diagrams, the effective action $W$ should be taken in the classical approximation:

$$W(\rho,\rho') = Y(\phi(\rho,\rho'),\phi^\dagger(\rho,\rho'),\rho,\rho'),$$

where the classical fields are determined from a pair of transcendental equations

$$\phi = ia\rho' e^{\phi^\dagger},\quad \phi^\dagger = ia\rho e^{\phi}.$$

Next we sum over all posible values of $n_i$, $n'_i$ and $c$. To sum over $n_i$ and $n'_i$ we present

$$C_{n,n'} = \oint \frac{dzdz'}{(2\pi i)^2 z^{n+1} z'^{n'+1}} iW(z,z').$$

Summations over $n_i$ and $n'_i$ in the integrand factorize in two factors

$$f_A(u_i) = \sum_{n_i}\frac{A!}{(A-n)!}\prod_{i=1}^{c} u_i^{n_i},\quad n = \sum_{1}^{c} n_i,$$

where $u_i = T_A(x_i)/z_i$ and a similar factor for the target B. We present each $u_i^{n_i}$ as

$$u^n = \frac{1}{un!}\int_0^\infty dt t^n e^{-t/u}$$

to find a sum in the integrand

$$\sum_{n_i} \frac{A!}{(A-n)!} \prod_{i=1}^{c} \frac{t_i^{n_i}}{n_i!} = \Big(1 + \sum_i^c t_i\Big)^A - \quad \text{zero} \quad \text{terms.} \tag{38}$$

The subtraction terms eliminate contributions from the $i$-th connected part when all $n_i = 0$. The right-hand side of Eq. (38) can be represented as a contour integral

$$\frac{A!}{2\pi i^{A+1}} \oint \frac{d\tau}{\tau^{A+1}} e^{i\tau} \prod_{i=1}^{c} \Big(e^{i\tau t_i} - 1\Big).$$

Integration over all $t_i$ gives

$$f_A(u_i) = \frac{A!}{2\pi i^{A+1}} \oint \frac{d\tau}{\tau^{A+1}} \prod_{i=1}^{c} \Big(\frac{1}{1 - i\tau u_i} - 1\Big).$$

The final expression for the amplitude is obtained after integrations over all $z_i$ and summation over the number of connected parts $c$, which are realized in a straightforward manner. Changing $\phi \to -u$ and $\phi^\dagger \to -v$ leads to the expression Eq. (6).

# References


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